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STOCHASTIC ANALYSIS OF DYNAMIC SYSTEMS CONTAINING FRACTIONAL DERIVATIVES

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1. INTRODUCTION

The accurate modelling of structural and flexible multibody systems requires an accurate modelling of the internal damping of the systems. It has been shown that fractional derivative models describe the frequency dependence of the structural damping very well [1–3]. Koeller [4] considered a fractional calculus model to obtain expressions for creep and relaxation functions for viscoelastic materials. Makris and Constantinou [5] presented a fractional derivative Maxwell model for viscous dampers and validated the model using experimental results. They also presented some analytical results for a fractionally damped single-degree-of-freedom system. Mainardi [6] presented the thermoelastic coupling in anelastic solids to account for a temperature fractional relaxation due to diffusion. Fractional-derivative-based techniques to model the damping behavior of materials and systems have been considered by Shen and Soong [7], Pritz [8], and Papoulia and Kelly [9] also. Enelund *et al.* [10] presented a fractional integral viscoelastic model and showed that this model has the same constitutive advantages as the fractional derivative viscoelastic model. Koh and Kelly [11], Makris and Constantinou [12], and Lee and Tsai [13] used fractional derivatives to model the seismic and vibration isolation. Lixia and Agrawal [14] presented a numerical scheme to solve a fractionally damped dynamic system. Recently, Mainardi [15] presented the role of fractional calculus in the continuum and statistical mechanics. This paper also includes the role of fractional calculus in the modelling of various viscoelastic systems.

Fractional derivative models has also been used to model the stability and control of viscoelastic structures. Skaar *et al.* [16] and Makroglou *et al.* [17] presented root locus analyses for one-dimensional controlled distributed structures whose damping behaviors were modelled with fractional order derivatives. Bagley and Calico [18] presented fractional order state equations for the control of viscoelastically damped structures. Their study showed that the feedback of fractional order time derivatives of structural displacements improves the system control performance. Mbodje *et al.* [19] presented a linear-quadratic optimal control of a rod whose damping mechanism was described in terms of fractional derivatives. Makris *et al.* [20] presented a general boundary-element formulation for the dynamic response of fluids whose viscoelastic behavior as modelled using real-valued parameters and fractional order derivatives. They showed that their analytical results agree with the experimental results. Fenander [21] presented a modal synthesis approach for fractionally damped systems. Suarez and Shokooh [22] and Rossikhin and Shitikova [23] presented applications of fractional operators to the analysis

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of damped vibrations of viscoelastic single-mass systems. Enelund and Josefson [24] presented a time-domain finite-element analysis of fractionally damped viscoelastic structures. Riewe [25, 26] demonstrated that damping forces, which have traditionally been excluded from classical theories of mechanics, could be included in classical mechanics using fractional derivatives. Riewe [25, 26] also provided a brief review of fractional calculus.

Several authors have used fractional calculus to model the statistical behavior of the systems. Mainardi [15] presented a fractional calculus approach to model the Brownian motion. His formulation led to a fractional Langevin, which he solved using the Laplace transform technique. Mainardi [15] also provided a brief review of the Brownian motion and the role of fractional models in this field. Spanos and Zeldin [27] presented a frequency-domain approach for the random vibration of fractionally damped systems. Recently, Agrawal [28] presented an analytical scheme for stochastic dynamic systems whose damping behavior is described by a fractional derivative of order $\frac{1}{2}$. In this approach, the eigenvector expansion method of Suarez and Shokooh [22] and the properties of the Laplace transforms of convolution integrals were used to obtain the desired results.

This brief review of fractional calculus and its role in the modelling of damping and stochastic behavior of discrete and continuous systems is by no means complete. Oldham and Spanier [29] wrote the first book dedicated to fractional calculus and its applications in applied mathematics, science, and engineering. Since then, there has been considerable research in this area. For the fundamentals and extensive review of fractional calculus and its applications, we refer the reader to references [29–32]. For the fundamentals of random vibrations of discrete and continuous systems containing integral derivatives, we refer the reader to references [33, 34].

In this paper, we present an analytical scheme for stochastic analysis of a single-degree-of-freedom spring–mass–damper system whose damping is described by a fractional derivative of order p/q, where both p and q are positive integers. We first present a Laplace transform approach to obtain a fractional Green's function and a Duhamel integral-type closed-form expression for the response of the system. The method presented is applicable to a deterministic as well as a random input. We then use these expressions to obtain the stochastic response of the system to a general class of random inputs. The response terms are then specialized for the white noise.

2. FRACTIONAL DYNAMIC MODEL AND THE GENERAL SOLUTION

The differential equation of a single-degree-of-freedom spring–mass–damper system whose damping characteristics are described by a fractional derivative of order p/q can be written as

$$mD^2x(t) + cD^{p/q}x(t) + kx(t) = f(t), \tag{1}$$

where m, c, and k represent the mass, damping, and stiffness coefficients, respectively, $f(t)$ is the externally applied force, and $D^{p/q}x(t)$ is the fractional derivative of order p/q of the displacement function $x(t)$. The definition of the fractional derivative will be given shortly. Depending on the nature of the input, $f(t)$ represents a deterministic force or a random process. For simplicity in the discussion to follow, the above system will be called a fractionally damped system of order p/q. When the damping force is equal to $cDx(t)$, the system will be called a damped system of order 1. Note that the dimension of c for a fractionally damped system of order p/q is not the same as the dimension for a damped system of order 1.

Equation (1) can also be written as

$$D^{nv}x(t) + aD^{pv}x(t) + bx(t) = f_m(t), \tag{2}$$

where $a = c/m, b = k/m, f_m(t) = f(t)/m, n = 2q$ and $v = 1/q$. Following Miller and Ross [30], we call equation (2) a fractional differential equation of order (n, q). When $q = 1$, equations (1) and (2) reduce to ordinary differential equations.

There are several definitions for the derivative of a fractional order [29–32]. In the discussion to follow, we consider the Riemann–Liouville and the Caputo fractional derivatives of order α, which are defined as

the Riemann–Liouville fractional derivative:

$$D^{\alpha}_{RL}x(t) = \frac{1}{\Gamma(k-\alpha)}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\int_0^t \frac{x(u)}{(t-u)^{\alpha+1-k}}\,\mathrm{d}u, \tag{3}$$

the Caputo fractional derivative:

$$D^{\alpha}_{C}x(t) = \frac{1}{\Gamma(k-\alpha)}\int_0^t \frac{x(u)}{(t-u)^{\alpha+1-k}}\,\mathrm{d}u, \tag{4}$$

where α is the order of the derivative, k is a positive integer such that $k-1<\alpha<k$, superscript (k) represents the kth derivative, Γ represents the Gamma function, and the subscripts RL and C represent the Riemann–Liouville and the Caputo fractional derivatives respectively. Theoretically, α can be any positive number. In this study, we consider $0<\alpha = p/q < 2$. For the Riemann–Liouville derivative (i.e., when $D^{\alpha} = D^{\alpha}_{RL}$), the application of the Laplace transform to equation (2) leads to

$$P(s^v)X(s) = F_m(s) + sA_1 + B_1, \tag{5}$$

where $X(s)$ and $F_m(s)$ are the Laplace transform of $x(t)$ and $f_m(t)$, and $P(z)$ is an indicial polynomial of order n defined as

$$P(z) = z^n + az^p + b, \tag{6}$$

the constants A_1 and B_1 are given as

$$A_1 = \begin{cases} x(0), & 0 < p/q \leqslant 1, \\ x(0) + aD^{(p/q-2)}_{RL}x(0), & 1 < p/q \leqslant 2 \end{cases}$$

and

$$B_1 = \dot{x}(0) + aD^{(p/q-1)}_{RL}.$$

As discussed in reference [30], the fractional derivative of $x(t)$ at $t = 0$ can be obtained using an extension of the initial value theorem for the Laplace transform. Taking the inverse Laplace transform of equation (5), we obtain

$$x(t) = x_{RL}(t; x(0), \dot{x}(0)) + \int_0^t G(t-\xi) f_m(\xi)\,\mathrm{d}\xi, \tag{7}$$

where x_{RL} is the response of the system corresponding to the initial state only, and

$$G(t) = L^{-1}\left[\frac{1}{P(s^{v})}\right]$$

is the fractional Green's function associated with the operator $P(D^{v})$ [30]. The subscript RL in equation (7) represents the solution corresponding to Riemann–Liouville derivative. Differentiating equation (7) with respect to time and using the properties of the Green function, we obtain

$$\dot{x}(t) = \dot{x}_{RL}(t; x(0), \dot{x}(0)) + \int_0^t \dot{G}(t-\xi) f_m(\xi)\,\mathrm{d}\xi. \tag{8}$$

For the Caputo derivative (i.e., when $D^{\alpha} = D^{\alpha}_{C}$), the application of the Laplace transform to equation (2) leads to

$$P(s^{v})X(s) = F_m(s) + A_2(s)x(0) + B_2(s)\dot{x}(0), \tag{9}$$

where

$$A_2(s) = s + as^{\alpha-1}$$

and

$$B_2(s) = \begin{cases} 1, & 0 < \alpha \leqslant 1, \\ 1 + s^{\alpha-2}, & 1 < \alpha \leqslant 2. \end{cases}$$

Taking the inverse Laplace transform of equation (9), we obtain

$$x(t) = x_C(t; x(0), \dot{x}(0)) + \int_0^t G(t-\xi) f_m(\xi)\,\mathrm{d}\xi, \tag{10}$$

where x_C represents the response of the system corresponding to the initial state only when the Caputo fractional derivative is considered. Differentiating equation (10) with respect to time and using the properties of the Green function, we obtain

$$\dot{x}(t) = \dot{x}_C(t; x(0), \dot{x}(0)) + \int_0^t \dot{G}(t-\xi) f_m(\xi)\,\mathrm{d}\xi. \tag{11}$$

Equations (7) [equation (10)] and (8) [equation (11)] represent general closed–form solutions for the position and the velocity for equation (1) or (2) corresponding to the Riemann–Liouville (the Caputo) fractional derivative. Note that these equations contain two parts each on their right-hand side, the force and the initial condition. The force parts represent the zero-state response, and the initial condition parts represent the zero-input response. These equations are similar to the Duhamel integral solution for a linear system. Therefore, they can be considered as the Duhamel integral formula for the dynamic system described by equation (1). Also note that the Green functions for the two fractional operators are the same.

3. STOCHASTIC ANALYSIS

Equations (7), (8), (10), and (11) are applicable for an arbitrary forcing function, and therefore, they are also applicable for a random input. For stochastic analysis, we consider

$f_m(t)$ as a Gaussian random process with a zero mean function and a specified correlation function $R(t, u)$, i.e.,

$$E[f_m(t)] = 0, \qquad R(t, u) = E[f_m(t) f_m(u)], \tag{12, 13}$$

where E is the expectation operator. The process $f_m(t)$ need not have a zero mean function. However, this assumption is made for simplicity. Applying E to equations (7), (8), (10) and (11), and using equation (12), we obtain the mean function for displacement and velocity processes as

$$\bar{x}(t) = E[x(t)] = x_{RL}(t; x(0), \dot{x}(0)), \qquad \bar{\dot{x}}(t) = E(\dot{x}(t) = \dot{x}_{RL}(t; x(0), \dot{x}(0)), \tag{14, 15}$$

for the Riemann–Liouville fractional derivative, and

$$\bar{x}(t) = E[x(t)] = x_C(t; x(0), \dot{x}(0)), \qquad \bar{\dot{x}}(t) = E(\dot{x}(t) = \dot{x}_C(t; x(0), \dot{x}(0)), \tag{16, 17}$$

for the Caputo fractional derivative. Using equations (7), (8), (10), (11), (14)–(17), it follows that for both derivatives

$$x(t) - E[x(t)] = \int_0^t G(t - \xi) f_m(\xi)\, \mathrm{d}\xi, \qquad \dot{x}(t) - E[\dot{x}(t)] = \int_0^t \dot{G}(t - \xi) f_m(\xi)\, \mathrm{d}\xi. \tag{18, 19}$$

Using equations (12), (13), (18) and (19), the variance and covariance functions are given as

$$E[(x(t) - E[x(t)])^2] = \int_0^t \int_0^t G(t - \xi_1) G(t - \xi_2) R(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2, \tag{20}$$

$$E[(\dot{x}(t) - E[\dot{x}(t)])^2] = \int_0^t \int_0^t \dot{G}(t - \xi_1) \dot{G}(t - \xi_2) R(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2, \tag{21}$$

$$E[(x(t) - E[x(t)])(\dot{x}(t) - E[\dot{x}(t)])] = \int_0^t \int_0^t G(t - \xi_1) \dot{G}(t - \xi_2) R(\xi_1, \xi_2)\, \mathrm{d}\xi_1\, \mathrm{d}\xi_2. \tag{22}$$

Note that in the above derivations, the initial conditions have been assumed to be deterministic. For random initial conditions, the above equations can be modified using an approach similar to the one for a stochastic damped system of order 1.

Equations (18)–(22) provide the stochastic response of the system for a general class of random input. As a special case, for white noise, $R(t, u) = q\delta(t - u)$; equations (20)–(22) after some algebraic manipulation reduce to

$$E[(x(t) - E[x(t)])^2] = q\int_0^t G^2(\xi)\, \mathrm{d}\xi, \qquad E[(\dot{x}(t) - E[\dot{x}(t)])^2] = q\int_0^t \dot{G}^2(\xi)\, \mathrm{d}\xi, \tag{23, 24}$$

$$E[(x(t) - E[x(t)])(\dot{x}(t) - E(\dot{x}(t)])] = q\int_0^t G(\xi)\, \dot{G}(\xi)\, \mathrm{d}\xi. \tag{25}$$

Closed form or numerical computation of equations (20)–(25) requires the knowledge of G and $\dot{G}$. A general method for computing these terms is discussed in Appendix A. Note that an eigenvector expansion approach for obtaining equations (20)–(25) for $\alpha = \frac{1}{2}$ was presented in reference [28]. The scheme presented is general and applicable to all fractional derivatives of positive rational order.

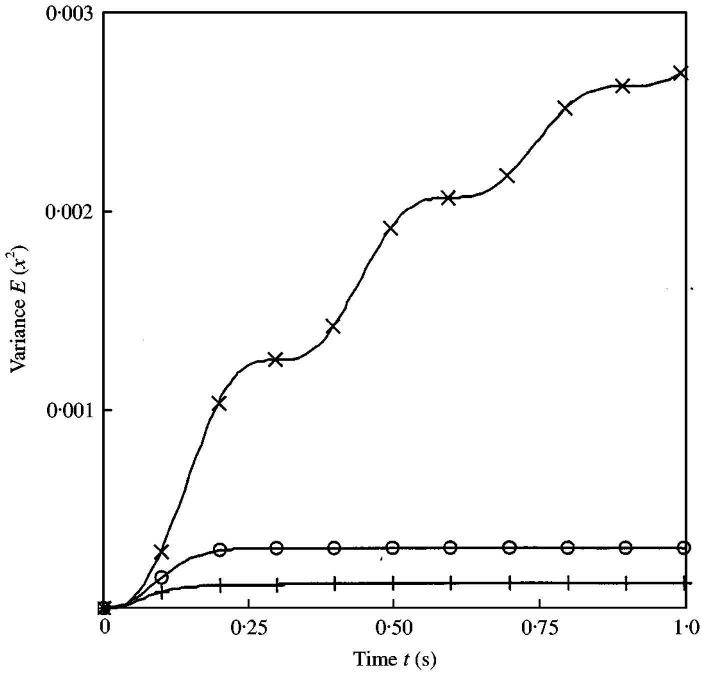

Figure 1. Variance function $E[x^2]$ as a function of time ($p/q = 2/3$): x, $\eta = 0{\cdot}05$; ○, $\eta = 0{\cdot}05$; $+$, $\eta = 1{\cdot}0$.

Equations (20)–(22) (or equations (23)–(25)) can be used to compute the stochastic response of the dynamic system. This approach is similar to the impulse function approach to find the stochastic response of a damped system of order 1. Another approach that utilizes frequency response functions for this purpose can be found in references [33, 34].

The above formulation presents expressions for covariance functions for only a single-degree-of-freedom system. For multi-degree-of-freedom systems, the formulation can be developed in a similar manner.

4. NUMERICAL RESULTS

In Appendix A, it is shown that the covariance functions for $\alpha = \frac{1}{2}$ obtained using the method presented here and the one in reference [28] are the same. In this section, we present some simulation results for the stochastic behavior of a fractionally damped system of order $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{4}{3}$ and compare them with those for a damped system of order 1. For simplicity in the discussion to follow, the following parameters are defined: $b = \omega^2 = k/m$ and $a = 2\eta\omega^{3/2} = c/m$, where ω is the natural frequency of the system, and the coefficient η is the damping ratio of the fractionally damped system of order $\frac{1}{2}$. The exponent $\frac{3}{2}$ on ω is introduced for consistency in dimensions. These definitions are given so that the parameters ω and η considered here are consistent with those defined in reference [28]. For numerical simulations, the following values are used: $\omega = 10$ and $\eta = 0{\cdot}05$, $0{\cdot}5$, and $1{\cdot}0$.

Figures 1–3 show the variance functions $E[x^2]$ and $E[v^2]$, and the covariance function $E[xv]$ as functions of time for $p/q = \frac{2}{3}$. Here $v = \dot{x}$. It is not surprising that as η increases, the values of these functions decrease. Note that these functions show some oscillations for

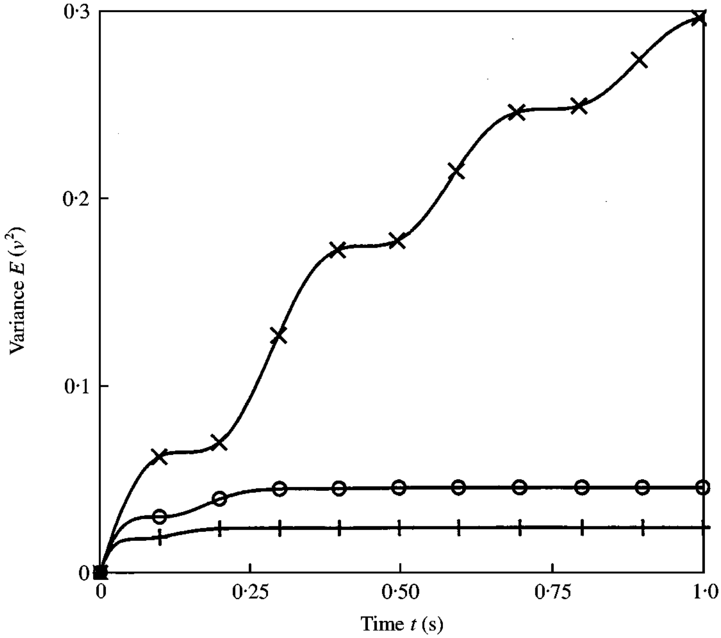

Figure 2. Variance function $E[v^2]$ as a function of time ($p/q = \frac{2}{3}$): x, $\eta = 0{\cdot}05$; ○, $\eta = 0{\cdot}05$; +, $\eta = 1{\cdot}0$.

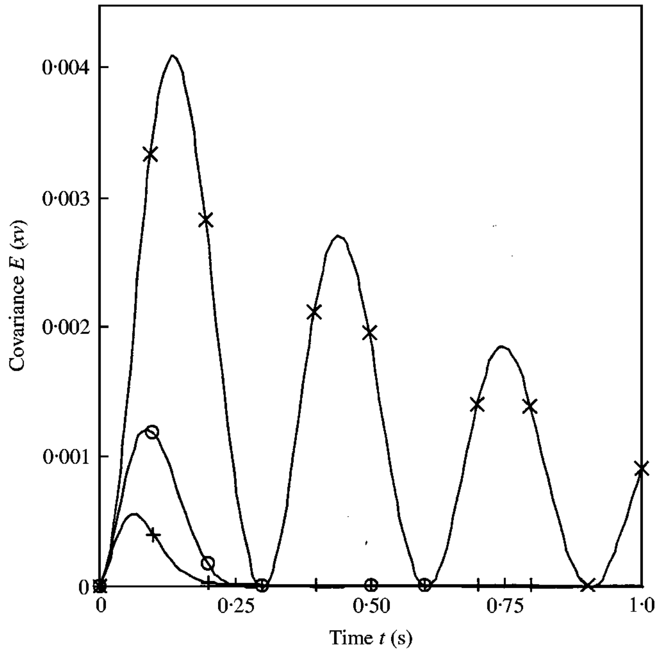

Figure 3. Covariance function $E[xv]$ as a function of time ($p/q = \frac{2}{3}$): x, $\eta = 0{\cdot}05$; ○, $\eta = 0{\cdot}05$; +, $\eta = 1{\cdot}0$.

$\eta = 0{\cdot}5$ and 1. As a matter of fact, the values of a and b for a first order system computed using $\eta = 0{\cdot}5$ leads to an over-damped system. This indicates that when p/q goes from 1 to 0, the damper acts more and more like a spring.

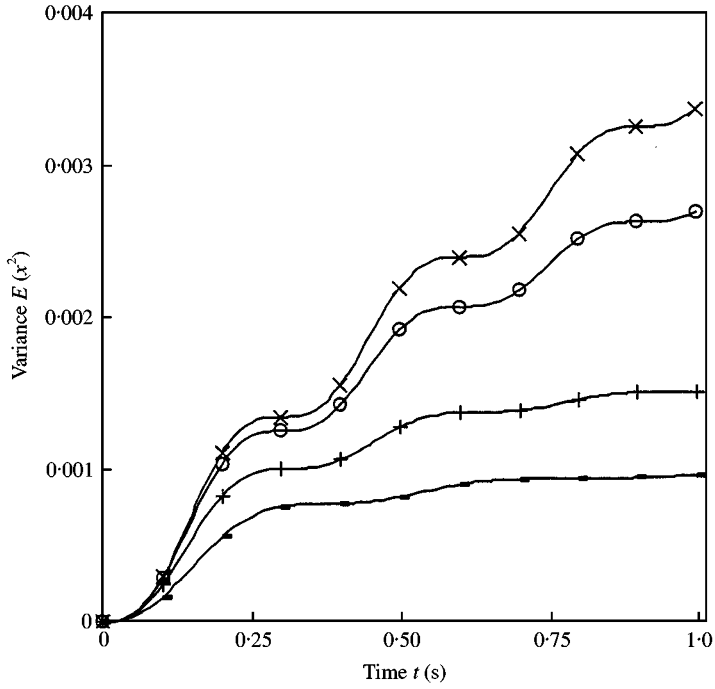

Figure 4. Comparison of $E[x^2]$ for fractional models ($\eta = 0{\cdot}05$): x, $p/q = \frac{1}{2}$; ○, $p/q = \frac{2}{3}$; +, $p/q = 1$; –, $p/q = \frac{4}{3}$.

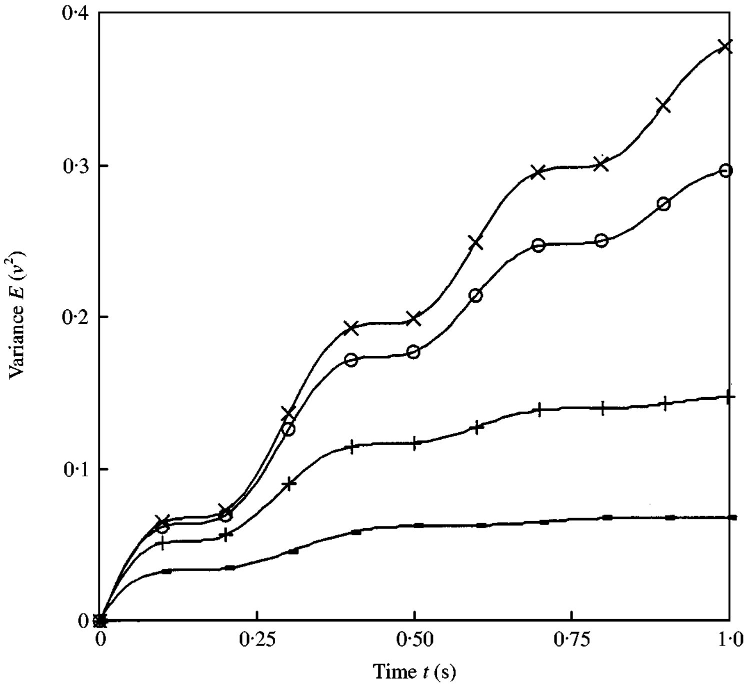

Figure 5. Comparison of $E[v^2]$ for fractional models ($\eta = 0{\cdot}05$): x, $p/q = \frac{1}{2}$; ○, $p/q = \frac{2}{3}$; +, $p/q = 1$; –, $p/q = \frac{4}{3}$.

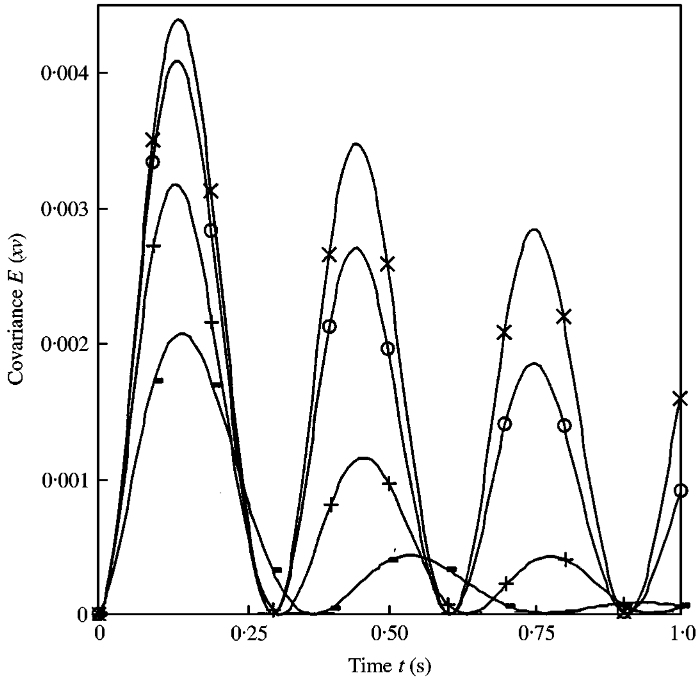

Figure 6. Comparison of $E[xv]$ for fractional models ($\eta = 0{\cdot}05$): x, $p/q = \frac{1}{2}$; O, $p/q = \frac{2}{3}$; +, $p/q = 1$; –, $p/q = \frac{4}{3}$.

To compare the results of fractionally damped systems of order $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{4}{3}$ with those of order 1, we take $\eta = 0{\cdot}05$. Figure 4 compares the variance function $E[x^2]$ for these systems. Corresponding results for the variance function $E[v^2]$ and the covariance function $E[xv]$ are shown in Figures 5 and 6. Note that as p/q goes from 1 to 2, the damper acts more and more like a mass.

5. CONCLUSIONS

A Laplace transform-based technique has been presented to obtain the Green function and the response of a single-degree-of-freedom spring–mass system whose damping behavior is described by a fractional derivative of order p/q. A Duhamel integral-type expression has been presented for the response of a fractionally damped dynamic system that may be subjected to deterministic or random input. These expressions were used to obtain the stochastic response of the system subjected to a general random input force, and as a special case, to a white noise. Two types of fractional derivatives, namely, Riemann–Liouville and Caputo, were considered. It was shown that both definitions of fractional derivatives lead to the same expressions for variance and covariance functions. Numerical results were presented to show the stochastic behavior of a fractionally damped system. The numerical results show that when the order of the derivative in the damping force term goes from 1 to 0 (1 to 2) a damper behaves more like a spring (mass).

REFERENCES

1. R. L. Bagley and P. J. Torvik 1983 *Journal of Rheology* **27**, 201–210. A theoretical basis for the application of fractional calculus to viscoelasticity.

2. R. L. BAGLEY and P. J. TORVIK 1983 *American Institute of Aeronautics and Astronautics Journal* **21**, 741–748. Fractional calculus—a different approach to the analysis of viscoelastically damped structures.
3. R. L. BAGLEY and P. J. TORVIK 1985 *American Institute of Aeronautics and Astronautics Journal* **23**, 918–925. Fractional calculus in the transient analysis of viscoelastically damped structures.
4. R. C. KOELLER 1984 *American Society of Mechanical Engineers Journal of Applied Mechanics* **51**, 299–307. Application of fractional calculus to the theory of viscoelasticity.
5. N. MAKRIS and M. C. CONSTANTINOU 1991 *Journal of Structural Engineering* **117**, 2708–2724. Fractional-derivative Maxwell model for viscous dampers.
6. F. MAINARDI 1994 *Journal of Alloys and Compounds* **211**, 534–538. Fractional relaxation in anelastic solids.
7. K. L. SHEN and T. T. SOONG 1995 *Journal of Engineering Mechanics* **121**, 694–701. Modeling of viscoelastic dampers for structural applications.
8. T. PRITZ 1996 *Journal of Sound and Vibration* **195**, 103–115. Analysis of four-parameter fractional derivative model of real solid materials.
9. K. D. PAPOULIA and J. M. KELLY 1997 *Journal of Engineering Materials and Technology, Transactions of the American Society of Mechanical Engineers* **119**, 292–297. Visco-hyperelastic model for filled rubbers used in vibration isolation.
10. M. ENELUND, A. FENANDER and P. OLSSON 1997 *American Institute of Aeronautics and Astronautics Journal* **35**, 1356–1362. Fractional integral formulation of constitutive equations of viscoelasticity.
11. C. G. KOH and J. M. KELLY 1990 *Earthquake Engineering and Structural Dynamics* **19**, 229–241. Application of fractional derivatives to seismic analysis of base-isolated models.
12. N. MAKRIS and M. C. CONSTANTINOU 1992 *Earthquake Engineering and Structural Dynamics* **21**, 649–664. Spring–viscous damper systems for combined seismic and vibration isolation.
13. H. H. LEE and C. S. TSAI 1994 *Computers and Structures* **50**, 111–121. Analytical model for viscoelastic dampers for seismic mitigation of structures.
14. Y. LIXIA and O. P. AGRAWAL 1998 *Proceedings of* 1998 *ASME Design Engineering Technical Conferences. Atlanta, GA*, September 13–16. A numerical scheme for dynamic systems containing fractional derivatives.
15. F. MAINARDI 1997 *in Fractals and Fractional Calculus in Continuum Mechanics* (A. Carpinteri and F. Mainardi, editors), 291–348. Wien, New York: Springer-Verlag. Fractional calculus: some basic problems in continuum and statistical mechanics.
16. S. B. SKAAR, A. N. MICHEL and R. K. MILLER 1988 *IEEE Transactions on Automatic Control* **33**, 348–357. Stability of viscoelastic control systems.
17. A. MAKROGLOU, R. K. MILLER and S. SKAAR 1994 *Journal of Guidance, Control, and Dynamics* **17**, 84–90. Computational results for a feedback control for a rotating viscoelastic beam.
18. R. L. BAGLEY and R. A. CALICO 1991 *Journal of Guidance, Control, and Dynamics* **14**, 304–311. Fractional order state equations for the control of viscoelastically damped structures.
19. B. MBODJE, C. MONTSENY, J. AUDOUNET and P. BENCHIMOL 1994 *The Proceedings of the Third IEEE Conference on Control Applications*. August 24–26, 1329–1333. Glasgow; The University of Strathclyde. Optimal control for fractionally damped flexible systems.
20. N. MAKRIS, G. F. DARGUSH and M. C. CONSTANTINOU 1993 *Journal of Engineering Mechanics* **119**, 1663–1679. Dynamic analysis of generalized viscoelastic fluids.
21. A. FENANDER 1996 *American Institute of Aeronautics and Astronautics Journal* **34**, 1051–1058. Modal synthesis when modeling damping by use of a fractional derivatives.
22. L. E. SUAREZ and A. SHOKOOH 1997 *American Society of Mechanical Engineers. Journal of Applied Mechanics* **64**, 629–635. An eigenvector expansion method for a solution of "motion containing fractional derivatives".
23. Y. A. ROSSIKHIN and M. V. SHITIKOVA 1997 *Journal of Sound and Vibration* **199**, 567–586. Application of fractional operators to the analysis of damped vibrations of viscoelastic single-mass systems.
24. M. ENELUND and B. L. JOSEFSON 1997 *American Institute of Aeronautics and Astronautics Journal* **35,** 1630–1637. Time-domain finite element analysis of viscoelastic structures with fractional derivatives constitutive relations.
25. F. RIEWE 1996 *Physical Review E* **53**, 1890–1899. Nonconservative Lagrangian and Hamiltonian mechanics.
26. F. RIEWE 1997 *Physical Review E* **55**, 3581–3592. Mechanics with fractional derivative.

27. P. D. SPANOS and B. A. ZELDIN 1997 *Journal of Engineering Mechanics* **123**, 290–292. Random vibration of systems with frequency-dependent parameters of fractional derivatives.
28. O. P. AGRAWAL 1999 *Proceedings of the 1999 American Society of Mechanical Engineers Design Engineering Technical Conferences Las Vegas, NV, September* 12–15, Paper No. DETC99/BIV-8238. An analytical scheme for stochastic dynamic systems containing fractional derivatives.
29. K. B. OLDHAM and J. SPANIER 1974 *The Fractional Calculus*. New York: Academic Press.
30. K. S. MILLER and B. ROSS 1993 *An Introduction to the Fractional Calculus and Fractional Differential Equations*. New York: John Wiley and Sons, Inc.
31. R. GORENFLO and F. MAINARDI 1997 in *Fractals and Fractional Calculus in Continuum Mechanics*. (A. Carpinteri and F. Mainardi, editors), 223–276. Wien, New York: Springer-Verlag. Fractional calculus: integral and differential equations of fractional order.
32. I. PODLUBNY 1999 *Fractional Differential Equations*. New York: Academic Press.
33. Y. K. LIN 1965 *Probabilistic Theory of Structural Dynamics*. New York: McGraw-Hill.
34. N. C. NIGAM 1983 *Introduction to Random Vibrations*. Cambridge: MIT Press.
35. M. R. SPIEGEL 1998 *Advanced Mathematics for Engineers and Scientists*. New York: McGraw-Hill.

APPENDIX A: EXPRESSION FOR THE FRACTIONAL GREEN'S FUNCTION

Both deterministic and stochastic analyses require the determination of the fractional Green's function $G(t)$ and its time derivative $\dot{G}(t)$. In this appendix, we present a general derivation of $G(t)$ and $\dot{G}(t)$ (see reference [30]) and then we specialize it for $q = 2$ and 3. We further show that the Green functions for $\alpha = \frac{1}{2}$ determined using this and the eigenvector expansion method [28] are the same.

As discussed in section 2, the fractional Green's function $G(t)$ is given as

$$G(t) = L^{-1}\left[\frac{1}{P(s^v)}\right], \tag{A1}$$

where $P(z)$ is the indicial polynomial of order n as defined in equation (6). Let λ_i, $i = 1, \ldots, n$, be the roots of this polynomial. Here we will consider that the roots are different. (For the case where two or more roots are equal, the readers are referred to reference [30].) Using the partial fraction expansion method, equation (A1) is written as

$$G(t) = L^{-1}\left[\sum_{i=1}^{n} \frac{A_i}{s^v - \lambda_i}\right] = \left[\sum_{i=1}^{n} \frac{A_i}{s^v - \lambda_i}\right], \tag{A2}$$

where

$$A_i = DP(\lambda_i), \quad i = 1, \ldots, n. \tag{A3}$$

As demonstrated in reference [30], $L^{-1}[1/(s^v - \lambda_i)]$ can be written as

$$L^{-1}\left[\frac{1}{s^v - \lambda_i}\right] = \sum_{j=1}^{q} \lambda^{j-1} E_t(jv - 1, \lambda_i^q), \tag{A4}$$

where $E_t(v, \lambda)$ is related to the incomplete gamma function by the relation

$$E_t(v, \lambda) = t^v e^{\lambda t} \gamma^*(v, \lambda t). \tag{A5}$$

Here γ^* is the incomplete gamma function. Using equations (A2) and (A4), we find that

$$G(t) = \sum_{i=1}^{n} A_i \sum_{j=1}^{q} \lambda_i^{j-1} E_t(jv - 1, \lambda_i^q). \tag{A6}$$

Differentiating equation (A6) with respect to time and using the properties of $E_t(v, \lambda)$ [30], we obtain

$$\dot{G}(t) = \sum_{i=1}^{n} A_i \sum_{j=1}^{q} \lambda_i^{j-1} E_t(jv - 2, \lambda_i^q). \tag{A7}$$

Equations (A6) and (A7) give the expressions for $G(t)$ and $\dot{G}(t)$ respectively. These equations contain terms that go to infinity at $t = 0$. These terms can be eliminated using the properties of the constants A_i, $i = 1, \ldots, n$. In particular, we consider $q = 2$ and 3. For $q = 2$, α can be $\frac{1}{2}$ or $\frac{3}{2}$, and for $q = 3$, α can be $\frac{1}{3}$, $\frac{2}{3}$, $\frac{4}{3}$, and $\frac{5}{3}$. For $q = 2$, we have (see Appendix C of Miller and Ross [30])

$$L^{-1}\left[\frac{1}{s^{1/2} - \lambda}\right] = \lambda^2 E_t(\tfrac{1}{2}, \lambda^2) + \lambda E_t(0, \lambda^2) + \frac{1}{\sqrt{\pi t}}. \tag{A8}$$

Note that the last term on the right-hand side (RHS) of equation (A8) goes to infinity as $t \to 0$. Using equations (A2) and (A8), $G(t)$ is given as

$$G(t) = \sum_{i=1}^{4} A_i [\lambda_i^2 E_t(\tfrac{1}{2}, \lambda_i^2) + \lambda_i E_t(0, \lambda_i^2)] + \frac{1}{\sqrt{\pi t}} \sum_{i=1}^{4} A_i. \tag{A9}$$

Using the properties of A_i, $i = 1, \ldots, 4$, it follows that the last term on the RHS of equation (A9) is zero (see Appendix A of Miller and Ross [30]). Using equations (A3) and (A9) and the properties of $E_t(v, \lambda)$ [30], it can be shown that

$$G(t) = \sum_{i=1}^{4} \frac{\lambda_i}{4\lambda_i^3 + a} e^{\lambda_i^2 t} [1 + \mathrm{Erf}(\lambda_i \sqrt{t})]. \tag{A10}$$

After some algebraic manipulation, it can be shown that the expression for $G(t)$ in equation (A10) is the same as that in reference [28].

For $q = 3$, we have (see Appendix C of Miller and Ross [30])

$$L^{-1}\left[\frac{1}{s^{1/3} - \lambda}\right] = \lambda^3 E_t(\tfrac{1}{3}, \lambda^3) + \lambda^4 E_t(\tfrac{2}{3}, \lambda^3) + \lambda^2 E_t(0, \lambda^3) + \frac{t^{-2/3}}{\Gamma(\frac{1}{3})} + \frac{t^{-1/3}}{\Gamma(\frac{2}{3})}. \tag{A11}$$

The last two terms on the RHS of equation (A11) tend to infinity as $t \to 0$. Using equations (A2) and (A11) and the properties of A_i, $i = 1, \ldots, 6$ (see Appendix A of Miller and Ross [30]), we obtain

$$G(t) = \sum_{i=1}^{6} A_i [\lambda_i^2 E_t(\tfrac{1}{3}, \lambda_i^3) + \lambda_i^4 E_t(\tfrac{2}{3}, \lambda_i^3) + \lambda_i^2 E_t(0, \lambda_i^3)]. \tag{A12}$$

Note that the terms that go to infinity as $t \to 0$ no longer exist in equation (A12). $\dot{G}(t)$ for $q = 2$ and 3 are obtained by taking the time derivative of equations (A10) and (A12).